

Brownian motion, random walks on trees, and harmonic measure on polynomial Julia sets

Nathaniel D. Emerson

2nd February 2008

Abstract

We consider the harmonic measure on a disconnected polynomial Julia set in terms of Brownian motion. We show that with probability one, the first point in the Julia set that a Brownian particle hits is a single-point component. Associated to the polynomial is a combinatorial model, the tree with dynamics. We define a random walk on the tree, which is a combinatorial version of Brownian motion in the plane. This random walk induces a measure on the tree, which is isomorphic to the harmonic measure on the Julia set.

1 Introduction

Let f be a polynomial of degree $d \geq 2$. The set of all points with a bounded orbit under f ,

$$\mathcal{K}_f = \left\{ z \in \mathbb{C} : \sup_n |f^n(z)| < \infty \right\},$$

is called *filled-in Julia set* of f , where f^n denotes the n^{th} iterate of f . The *Julia set* of f is the topological boundary of \mathcal{K}_f : $\mathcal{J}_f = \partial\mathcal{K}_f$. We will only consider f with \mathcal{K}_f disconnected. By a classical result of Fatou and Julia, this means at least one critical point of f has unbounded orbit.

Imagine a particle moving on the Riemann sphere according to the laws of Brownian motion [Do]. Suppose the particle starts at the point at infinity and let $Z(t)$ denote the position of the particle at time t . We call $Z(t)$ a *Brownian path*. Following A. Lopes [Lo] and S. Lalley [La], we consider the interaction of a Brownian particle with a polynomial Julia set. We say an event *almost surely* (*a.s.*) occurs if the probability of it occurring is 1. A Brownian path almost surely enters \mathcal{K}_f in a finite amount of time. Let $t_0 = \inf \{t : Z(t) \in \mathcal{K}_f\}$. We call t_0 the *first entry time* of Z into \mathcal{K}_f , and $Z(t_0)$ the *first entry point*.

Brownian motion induces a measure on \mathcal{K}_f , which we call ω_f , the *harmonic measure* of f [Do]. For X a measurable subset of \mathcal{K}_f , $\omega_f(X)$ is the probability that the first entry point of a Brownian path lies in X . The harmonic measure of f , is a Borel probability

measure defined on \mathcal{K}_f . It is f -invariant, f -ergodic and strongly mixing. Roughly, the harmonic measure describes the one-dimensional structure of \mathcal{K}_f . A useful feature of the harmonic measure is that it has a variety of equivalent definitions, see Theorem 2.1. We will consider harmonic measure in terms of Brownian particles for intuitive purposes [La]. For technical purposes, we will define it in terms of landing external rays [A].

We consider the interaction of disconnected polynomial Julia sets and Brownian particles. We consider disconnected polynomial Julia sets. In particular, we consider Julia sets that have connected components which are not points, which we refer to as *island components*. We call a single-point component a *singleton*. Note the island components are clearly visible in Figure 3.2. A disconnected polynomial Julia set will always have uncountably many singletons. So, the the structure of the Julia sets that we consider is something like an island chain surrounded by a barrier reef. The island components are larger than singletons in a topological sense. For instance they have positive diameter. They are also larger in terms of Brownian particles, a Brownian path will almost surely visit a given island component.

Theorem A. *If f is a polynomial with a disconnected Julia set, then the first entry point of a Brownian path into \mathcal{K}_f is almost surely a singleton.*

So in terms of Brownian particles, island components are no larger than singletons. This result implies a number of facts about the structure of Julia sets that have countably many connected components that are not points.

We give an estimate on how quickly the harmonic measure decreases with respect to equipotentials of Green's function (Theorem 3.3). We show that the harmonic measure always decreases exponentially. It follows that the harmonic measure of any component of a disconnected polynomial Julia set is zero.

The major tool in this paper is the combinatorial system of a *tree with dynamics* [E]. Associated to a polynomial with a disconnected Julia set is a canonical tree with dynamics. The tree with dynamics is a discrete model for the dynamics of such a polynomial. It captures many important facets of the dynamics of a polynomial, but is easier to work with than the polynomial itself. We construct the tree by decomposing the basin of attraction of the point at infinity into conformal annuli using Green's function. These annuli have a natural tree structure, which is compatible with the dynamics. The polynomial has a well-defined degree on each annulus. We associate each annulus to a vertex of the tree. We obtain a countable, rooted tree \mathcal{T} , a map $F : \mathcal{T} \rightarrow \mathcal{T}$, and a degree function $\deg : \mathcal{T} \rightarrow \mathbb{Z}^+$. We define *the combinatorial harmonic measure* on \mathcal{T} . We put a point-mass at the root of \mathcal{T} . We distribute the mass of a vertex to its pre-images under F , weighting by the degree of a pre-image. We show that the harmonic measure of a component of the Julia set can be estimated by the combinatorial harmonic measure on the tree. This result allows us to prove our other main theorem: the combinatorial harmonic measure is a model for the harmonic measure in the plane.

Theorem B. *If f is a polynomial with a disconnected Julia set, then the combinatorial harmonic measure on the tree with dynamics of f is isomorphic to the harmonic measure on the Julia set of f .*

This allows us to use techniques from the field of discrete potential theory to answer questions about the harmonic measure. Harmonic measure in the plane can be defined by Brownian motion. The combinatorial harmonic measure can be defined a random walk on the tree.

The rest of this paper is organized as follows.

In Section 2, we give some background on potential theory. We describe the decomposition of the plane using Green's function. While this is a standard technique, there are some subtle points we later use. We recall some facts about the harmonic measure.

We state our results for Julia sets in Section 3. Especially an estimate on the rate of decrease of harmonic measure. We discuss various consequences of this result. To show Theorem A, we use a recent result of W. Qiu and Y. Yin [QY].

We describe the tree with dynamics of a polynomial, in Section 4. We define the combinatorial harmonic measure on the tree with dynamics. We give the combinatorial results that imply Theorem A. We show that the combinatorial harmonic measure is isomorphic to the harmonic measure on the filled-in Julia set, proving Theorem B.

While completing this paper, the author learned that L. DeMarco and C. McMullen had independently obtained many of the same results, in particular Theorems A and B [DeMc].

2 Background

We will consider the interaction of holomorphic dynamics and potential theory. We assume basic familiarity with holomorphic dynamics [S]. We use two objects from potential theory: Green's function and harmonic measure. The book of N. Steinmetz has a short introduction to Green's function and harmonic measure on polynomial Julia sets [S]. T. Ransford has written a very readable introduction to potential theory, which includes a section on polynomial dynamics [R]. The work of J. Doob gives a more complete account of potential theory, and covers Brownian motion in detail [Do]. The paper of S. Lalley is a good introduction to the particulars of Brownian motion and Julia sets of rational functions [La].

We give the details of the dynamical decomposition of the plane. Following Branner and Hubbard, we use equipotentials of Green's function of a polynomial to decompose the plane into conformal annuli [B]. These annuli have a natural tree structure, which is compatible with the dynamics. So we can associate a polynomial to the combinatorial system of tree with dynamics due to R. Pérez-Marco [E].

We then recall some facts about the harmonic measure of a polynomial Julia set. We consider harmonic measure in terms of landing external rays [A] and Brownian motion [La]. We give a variety of equivalent definitions of harmonic measure. We discuss subsets that are *shielded* from the harmonic measure—a phenomenon that we later show occurs in polynomial Julia sets.

2.1 The Dynamical Decomposition of the Plane

We define an *annulus* as a subset of the complex plane that is conformally equivalent to a set of the form $\{z \in \mathbb{C} : r_1 < |z| < r_2\}$, for some r_1, r_2 with $0 \leq r_1 < r_2 \leq \infty$. We say a set S is *nested* inside an annulus A , if S is contained in the bounded components of $\mathbb{C} \setminus A$. For an annulus A , we define the *filled-in annulus*

$$P(A) = A \cup \{\text{bounded components of } \mathbb{C} \setminus A\}.$$

Observe that $P(A)$ is an open topological disk.

For the remainder of this paper, let f be a polynomial of degree $d \geq 2$ with disconnected Julia set. Let g denote Green's Function of f . The functional equation $g(f) = d \cdot g$ is satisfied by f and g . We use g to define the dynamic decomposition of the basin of attraction of infinity for f .

An *equipotential* is a level set of g ; $\{z \in \mathbb{C} : g(z) = \lambda > 0\}$. The critical points of g are the critical points of f and the pre-images of critical points of f . We distinguish all equipotentials of g that contain a critical point of g or an image under f of a critical point of g . There are countably many such equipotentials, say $\{E_l\}_{l \in \mathbb{Z}}$. We index them so that $g|E_l < g|E_{l-1}$, E_l is a Jordan curve for $l \leq 0$, and E_1 is not a Jordan curve (so it contains a subset homeomorphic to a figure-8). Let H be the number of orbits of $\{E_l\}_{l \in \mathbb{Z}}$ under f . If f has e distinct critical points that escape to infinity, then $H \leq e$. It is possible that $H < e$, if f has two critical points c_1 and c_2 such that $g(c_1) = d^n g(c_2)$ for some $n \in \mathbb{Z}$. From the functional equation and the indexing of E_l , it follows that $f(E_l) = E_{l-H}$ for all l .

Define $U_l = \{z : g|E_l > g(z) > g|E_{l+1}\}$. For $l \leq 0$, U_l is a single annulus. For all l , U_l is open and consists of the disjoint union of finitely many annuli $A_{l,i}$. We call each of the $A_{l,i}$ an *annulus of f at level l* . The closure of a filled-in annulus, $\overline{P}(A_{l,i})$, is called a *puzzle piece of f at depth l* [B]. A sequence $(A_l)_{l=0}^\infty$ of annuli of f is called *nested*, if $A_l \subset U_l$ and A_{l+1} is nested inside A_l for all l . The intersection of the filled-in annuli $\bigcap_{l=0}^\infty P(A_l)$ from nested sequence is a connected component of \mathcal{K}_f .

We code the dynamics of a polynomial with a disconnected Julia set by the combinatorial system of a *tree with dynamics* [E]. We use $\{A_{l,i}\}$, the annuli of f , to form a tree \mathcal{T} by associating each $A_{l,i}$ to a vertex $a_{l,i}$. We define a map $\tau : \mathcal{T} \rightarrow \{A_{l,i}\}$ by $\tau(a_{l,i}) = A_{l,i}$. We declare that there is an edge between $a_{l,i}$ and $a_{l-1,j}$ if $A_{l,i}$ is nested inside $A_{l-1,j}$. From the functional equation $g(f) = d \cdot g$, we can show that the image of any annulus of f is another annulus of f . That is, for any $A_{l,i}$, we have $f(A_{l,i}) = A_{l-H,j}$ for some j . So the dynamics are compatible with the tree structure, and we define $F : \mathcal{T} \rightarrow \mathcal{T}$ by

$$F(a) = b \quad \text{if} \quad f(\tau(a)) = \tau(b).$$

Note that τ conjugates F to f , that is $\tau(F(a)) = f(\tau(a))$. We define the *degree* of each vertex $\deg a_{l,i}$ as the topological degree of $f|A_{l,i}$. We call the triple $\langle \mathcal{T}, F, \deg \rangle$ the *tree with dynamics* of f . When drawing trees, we show a vertex of degree 1 by \bullet , and a vertex of degree $D > 1$ by \textcircled{D} .

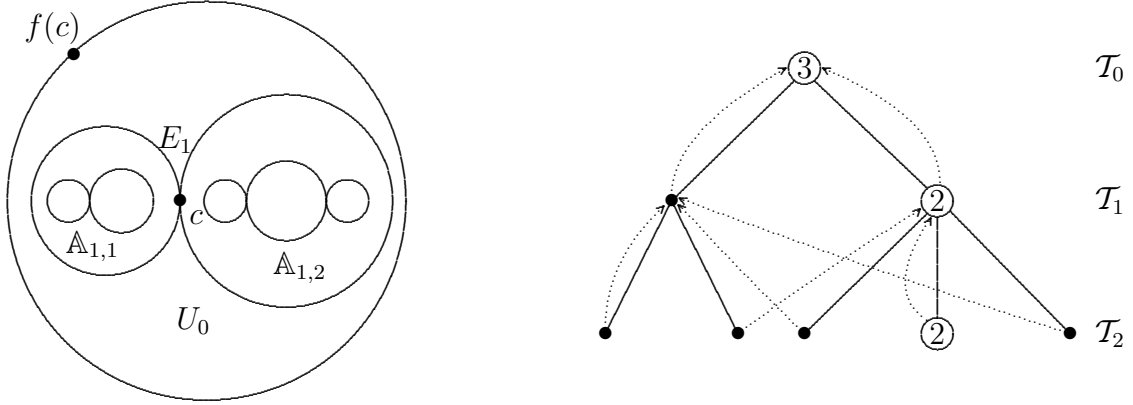


Figure 1: Equipotentials of a cubic polynomial with one escaping critical point, c (left). Its tree with dynamics, with $H = 1$ (right).

2.2 Harmonic Measure

There is a Borel probability measure on \mathcal{K}_f , the *harmonic measure* ω_f [R]. In fact, the support of ω_f is always contained in \mathcal{J}_f . So whether one considers ω_f a measure on \mathcal{K}_f or \mathcal{J}_f is a matter of preference. The harmonic measure is f -invariant, ergodic, and non-atomic. The harmonic measure is always mutually singular to two-dimensional Lebesgue measure [Ø1]. The support of ω_f has Hausdorff dimension at most 1 [JW]. The harmonic measure depends only on the topology of \mathcal{K}_f , and not the conformal structure.

For technical purposes we will define the harmonic measure in terms of landing external rays. This is a special case of the Green's measure [BreC]. For intuitive purposes, we will consider the harmonic measure in terms of Brownian motion [La].

A *Green's line* is an orthogonal trajectory to the equipotentials of Green's function. For a polynomial of degree at least 2, each Green's line can be canonically identified with a point in the circle at infinity, that is the set of asymptotic directions in the plane. An *external ray*, \mathcal{R}_θ , is a Green's line labelled with an angle θ from the circle at infinity $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. We say an external ray is *smooth* if it does not contain a critical point of Green's function. All but countably many external rays are smooth. Hence in terms of Lebesgue measure on the circle, almost all external rays are smooth. A smooth external ray intersects each equipotential of Green's function in a unique point, so we can parameterize an smooth ray by potential. For $\lambda > 0$, let $\mathcal{R}_\theta(\lambda)$ be the the unique point in $\mathcal{R}_\theta \cap \{g = \lambda\}$. We say that a smooth ray \mathcal{R}_θ *lands* at $z \in \mathcal{K}_f$ if $\lim_{\lambda \rightarrow 0^+} \mathcal{R}_\theta(\lambda) = z$. Almost every external ray lands. For X a measurable subset of \mathcal{K}_f , we have $\omega_f(X) = \text{Leb}_1(\{\theta : \mathcal{R}_\theta \text{ lands at } X\})$ [BreC, A], where Leb_1 denotes the normalized Lebesgue measure on the unit circle \mathbb{T} .

There are a variety of characterizations of the harmonic measure which we recall now.

Theorem 2.1. *If \mathcal{K} is the filled-in Julia set of a polynomial of degree at least 2, then the following measures on \mathcal{K} are equal:*

- a. the harmonic measure [R];

- b. the equilibrium measure $[R]$;
- c. the Green's measure $[A]$;
- d. the hitting measure of Brownian motion $[Do]$;
- e. the Brolin measure $[Bro]$.

In fact, for K a compact subset of \mathbb{C} with positive capacity, a–d are always equal.

We briefly consider the harmonic measure of more general compact subsets of the plane. If $X \subset \mathbb{C}$ is compact with positive capacity, the harmonic measure ω_X (with a pole at the point at infinity) is defined on X $[R]$. If X is a rectifiable curve, then ω_X is just the normalized one-dimensional Lebesgue measure on X . For instance, if X is a circle and A an arc of angle θ , then $\omega_X(A) = \theta/2\pi$. If X is a square and S one of its sides, $\omega_X(S) = 1/4$. The harmonic measure of $C \subset X$ depends not only on the intrinsic properties of C , but on how C is embedded in X . If $A \cup B$ is a partition of X , let us say A shields B if the first entry point of a Brownian particle almost surely lies in A . That is, if $\omega_X(A) = 1$. There is an intuitive explanation for shielding in terms of Brownian motion. Refer to Figure 2.2. Let S be a square and C be a circle. Take X as S enclosed by C . For this X , $\omega_X(S) = 0$, since a Brownian path must intersect the circle before it hits S . Now let Y be S enclosed by C and remove an arc of angle θ from C . Then $\omega_Y(S) \leq \theta/2\pi$, since a Brownian path whose first entry point is in S , must first pass through the gap in C . Now imagine we form Z by removing countably many arcs from C , so that what remains is a Cantor set of length L , where $0 \leq L < 1$. It follows that $\omega_Z(S) \leq 1 - L$.

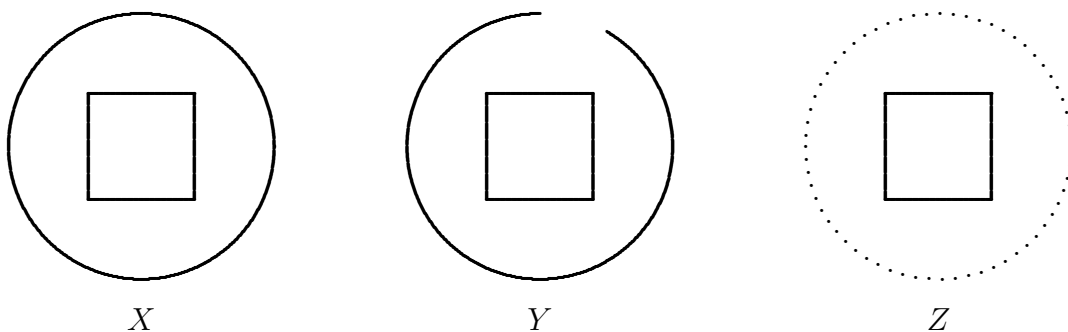


Figure 2: A subset shielded from the harmonic measure.

We show that shielding occurs for Julia sets. If \mathcal{K}_f is a disconnected polynomial Julia set with a component that is not a singleton, then the singletons of \mathcal{K}_f shield the island components, see Theorem A. This is similar to the situation in the set Z above. In light of Theorem 2.1, the islands components are also shielded from external rays.

3 Results for Julia sets

In this section, we give results for disconnected polynomial Julia sets. We only consider disconnected Julia sets. We will assume two results from Section 4, Theorem 4.12 and Lemma 4.15.

First, we will consider arbitrary disconnected Julia sets. We extend the harmonic measure to annuli of f . We show that the harmonic measure of an annulus decreases as an exponentially with the level of the annulus.

We then restrict our attention to disconnected Julia sets with a component that is not a point. H. Brolin gave the first example of such a Julia set [Bro, p.137–138]. B. Branner and J. Hubbard showed if the component containing a critical point of f is periodic, then that component and all of its pre-images are not singletons [BH, Thm. 5.3]. Recently, Qiu and Yin announced in a pre-print that \mathcal{K}_f is a Cantor set unless it has a periodic component containing a critical point [QY, Main Thm.].

3.1 Rate of decrease for the harmonic measure

We now extend the notion of harmonic measure to an annulus of f . By slight abuse of notation, we use ω_f to represent this notion.

Definition 3.1. *Let A be an annulus of f . Define*

$$\omega_f(A) = \text{Leb}_1 \{ \theta : \mathcal{R}_\theta \cap A \neq \emptyset \}.$$

We give an estimate on the harmonic measure of components of the Julia set in terms of $\omega_f(A)$.

Lemma 3.2. *Let f be a polynomial of degree ≥ 2 . Let K be a component of \mathcal{K}_f . Let $\{A_l\}_{l=0}^\infty$ be the unique nested sequence of annuli of f such that $K = \bigcap_{l=0}^\infty P(A_l)$. Then*

$$\omega_f(K) \leq \lim_{l \rightarrow \infty} \omega_f(A_l).$$

Proof. By Theorem 2.1, $\omega_f(K)$ is the measure of external rays that land on K . For each l , K is nested inside A_l and the boundary of A_l is contained in two equipotentials. Hence, any ray that lands on K must also intersect A_l . Thus, $\omega_f(K) \leq \omega_f(A_l)$ for each l . Since A_{l+1} is nested inside A_l , we have $\omega_f(A_{l+1}) \leq \omega_f(A_l)$. Taking a limit finishes the lemma. \square

Lemma 3.2 is an important result for proving Theorem A. It allows us to estimate $\omega_f(A_l)$, instead of computing $\omega_f(K)$ directly. We can easily transfer these estimates to the tree with dynamics. We show $\omega_f(A_l)$ decreases exponentially with l . This follows from the analogous combinatorial result, Theorem 4.12 and Lemma 4.15. Let $\lceil \cdot \rceil$ denote the ceiling function.

Theorem 3.3. *Let f be a polynomial of degree $d \geq 2$ with a disconnected Julia set. Let $D = 1 + M$, where M is the maximum of the multiplicities of the non-escaping critical points of f . Let H be the number of orbits of escaping critical points under f . There exists a constant $c_0 > 0$, such that if A is an annulus of f at level l , then*

$$\omega_f(A) \leq c_0 \left(\frac{D}{d} \right)^{\lceil l/H \rceil}.$$

We restate the above theorem in terms of escaping critical points. This is of interest if one considers a polynomial where the number or multiplicity of escaping critical points is known, but H is not. For instance, in the case of a polynomial from some escape locus in parameter space [BH].

Corollary 3.4. *Let f be a polynomial of degree $d \geq 2$. Let D , H , and A be the same as in Theorem 3.3. Let e be the number of distinct critical points of f that escape to infinity. Let m be the number of critical points of f , counted by multiplicity, that escape to infinity. Then*

$$\omega_f(A) \leq c_0 \left(\frac{D}{d} \right)^{\lceil l/H \rceil} \leq c_0 \left(\frac{D}{d} \right)^{\lceil l/e \rceil} \leq c_0 \left(\frac{D}{d} \right)^{\lceil l/m \rceil}.$$

Proof. We have $H \leq e \leq m$, so the last two inequalities are easily verified. \square

It follows that the harmonic measure of a component of a disconnected Julia set is 0.

Corollary 3.5. *Proof.* For every $l \geq 0$, there is a unique annulus A_l of f at level l such that K is nested inside A_l . Combining Lemma 3.2 and Theorem 3.3, we obtain

$$\omega_f(K) \leq \lim_{l \rightarrow \infty} \omega_f(A_l) \leq \lim_{l \rightarrow \infty} c_0 \left(\frac{D}{d} \right)^{\lceil l/H \rceil}.$$

Since \mathcal{K}_f is disconnected, we have $D < d$, so the right hand side tends to 0 as l approaches ∞ . \square

That is to say, no component of \mathcal{K}_f is charged by ω_f . Note that one could prove this by f -invariance of ω_f .

3.2 Julia sets with island components

For the rest of the section we assume that the Julia set has a component that is not a singleton. If K is a component of \mathcal{K}_f that is not a singleton, it will have positive capacity. Nonetheless, it will not be charged by ω_f .

We partition the Julia set into singletons and non-singletons. Let $K(z)$ denote the connected component of a point z in \mathcal{K}_f . Define

$$\mathcal{K}_f^0 = \{z : K(z) = \{z\}\} \quad \text{and} \quad \mathcal{K}_f^1 = \{z : K(z) \neq \{z\}\}.$$

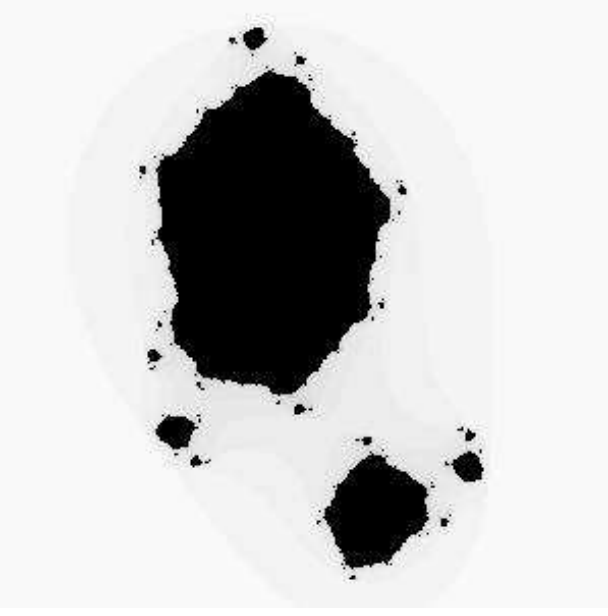


Figure 3: A disconnected polynomial Julia set with island components.

In Figure 3.2, the large “islands” are the components of \mathcal{K}_f^1 . The points can be thought of as \mathcal{K}_f^0 . We study the harmonic measure and this partition. Since the partition is f -invariant and ω_f is ergodic, one of these sets must have harmonic measure zero.

Any component of \mathcal{K}_f^1 has positive diameter. Thus, in a topological sense, components of \mathcal{K}_f^1 are larger than components of \mathcal{K}_f^0 . We ask which of \mathcal{K}_f^0 and \mathcal{K}_f^1 is larger in the sense of harmonic measure? This is not just a question of intrinsic properties of the components, but depends on how \mathcal{K}_f^0 and \mathcal{K}_f^1 are embedded in \mathcal{K}_f .

In terms of Brownian motion, a natural way to partition the Julia set is into those components that a Brownian path almost surely does not enter, and those components that it almost surely does enter. This is the same partition of the Julia set given above: \mathcal{K}_f^0 is the points of \mathcal{K}_f that lie in components that the path almost surely does not enter, and \mathcal{K}_f^1 is the points of \mathcal{K}_f that lie in components that the path almost surely enters:

$$\mathcal{K}_f^0 = \{z : Z(t) \text{ a.s. does not enter } \mathcal{K}_f(z)\} \quad \text{and} \quad \mathcal{K}_f^1 = \{z : Z(t) \text{ a.s. enters } \mathcal{K}_f(z)\}.$$

Hence, the components of \mathcal{K}_f^1 are “larger” than the components of \mathcal{K}_f^0 for Brownian motion in some sense. However, we prove the probability that the first entry point of a Brownian path lies in any given component is zero. So for Brownian motion, one could say that no component of \mathcal{K}_f is larger than any other. The explanation for this apparently contradictory result is that \mathcal{K}_f^0 shields \mathcal{K}_f^1 from Brownian particles. We restate Theorem A, in terms of our partition.

Theorem A. *If \mathcal{K}_f is a disconnected polynomial Julia set, then the first entry point of a Brownian path almost surely lies in \mathcal{K}_f^0 .*

Proof. By Corollary 3.5, the measure of every component of \mathcal{K}_f^1 is zero. Qiu and Yin announced that all but at most countably many components of \mathcal{K}_f are singletons [QY, p. 26]. It follows that the harmonic measure of the non-singletons of \mathcal{K}_f is zero. \square

That is, a Brownian path almost surely hits a point of \mathcal{K}_f^0 before it hits a point of \mathcal{K}_f^1 . Although individually the points of \mathcal{K}_f^0 are negligible, taken together they form an impenetrable barrier. Intuitively, the singleton components are rocks that prevent ships from landing on the islands components.

Theorem A also tells us where external rays land in \mathcal{K}_f . Intuitively, we can say that \mathcal{K}_f^0 shields \mathcal{K}_f^1 from external rays.

Corollary 3.6. *If \mathcal{K}_f is a disconnected polynomial Julia set, then almost every external ray lands on a singleton of \mathcal{K}_f .*

G. Levin and F. Przytycki have shown that for \mathcal{K}_f disconnected, if K is a periodic or pre-periodic component of \mathcal{K}_f , then some external ray land on K and every accessible point $z \in K$ is accessible via an external ray [LP,]. Corollary 3.6 can be thought of as a measure theoretic dual of their result. Topologically components of \mathcal{K}_f are visible, however they are shielded from the harmonic measure. \square

4 The Tree with Dynamics

This section is the technical heart of this paper. We work with the tree with dynamics. First, we recall some properties of the tree with dynamics. We then define a version of harmonic measure on the tree and show that it agrees with the harmonic measure on the annuli of f . We extend the measure to the boundary of the tree. We show that the measure on the boundary of the tree is isomorphic to the harmonic measure in the plane. We use the measure on the tree to define a random walk. We note the equivalence between random walks on the tree and Brownian motion in the plane

4.1 Preliminaries

We recall some notation from Section 2. We decompose the basin of attraction of infinity into open sets $\{U_l\}_{l \in \mathbb{Z}}$, bounded by equipotentials of Green's function. For each l , $U_l = \bigcup_{i=1}^n A_{l,i}$, where each $A_{l,i}$ is an annulus of f . We form the tree with dynamics by associating a vertex of \mathcal{T} to an annulus of f . The map $\tau : \mathcal{T} \rightarrow \{A_{l,i}\}$ witnesses this association. The tree with dynamics is a triple $\langle \mathcal{T}, F, \deg \rangle$, where \mathcal{T} is a tree, $F : \mathcal{T} \rightarrow \mathcal{T}$ is the dynamics, and $\deg : \mathcal{T} \rightarrow \mathbb{Z}^+$ is a degree.

We briefly state some properties of a tree with dynamics without proof. A more complete discussion can be found in a previous paper of the author [E]. The tree \mathcal{T} is a countable. That is, it is a countable graph with no non-trivial circuits. It can naturally be decomposed into levels.

Definitions 4.1. For $l \in \mathbb{Z}$, define $\mathcal{T}_l = \tau^{-1}(U_l)$. For $l \leq 0$, \mathcal{T}_l consists of a single vertex, say $\mathcal{T}_l = \{a_l\}$. We call a_0 the root of \mathcal{T} , and $\{a_{-l}\}_{l=1}^\infty$ the extended root of \mathcal{T} . Let $a \in \mathcal{T}_l$ for some $l \in \mathbb{Z}$. We call the unique vertex in \mathcal{T}_{l-1} that is adjacent to a the parent of a , and denote it by a^P . Any vertex in \mathcal{T}_{l+1} that is adjacent to a is called a child of a , and denoted by a^C .

Our convention in drawing trees is that a parent is above its children, as in a genealogic tree. So a^P is above a and any a^C is below a . We generally denote the set of all children of a by $\{a^C\}$. When it is necessary to distinguish among the children of a we use the notation $\{a^{C_i}\}$. The structure of the extended root is trivial. Its main purpose is to ensure that all iterates of the dynamics are defined.

Lemma 4.2. *The tree satisfies the following properties:*

- a. For any $a \in \mathcal{T}$, there is at least 1 child of a . That is, \mathcal{T} has no leaves.
- b. For any $a \in \mathcal{T}$, there are only finitely many children of a . That is, \mathcal{T} is locally finite.
- c. The root of \mathcal{T} , a_0 , has at least 2 children.

Lemma 4.3. *The dynamics satisfy the following properties:*

- a. The dynamics are children preserving. For any $a \in \mathcal{T}$, the image of a child of a is a child of $F(a)$. Symbolically, $F(a^C) = F(a)^C$.
- b. There exists $H \in \mathbb{Z}^+$ such that if $a \in \mathcal{T}_l$, then $F(a) \in \mathcal{T}_{l-H}$.
- c. The dynamics are locally a branched cover of \mathcal{T} . For any $a \in \mathcal{T}$, for each child $F(a)^C$ of $F(a)$ we have

$$\sum_{\{F(a^{C_i})=F(a)^C\}} \deg a^{C_i} = \deg a,$$

We refer to this as the local cover property.

Lemma 4.4. *The degree function satisfies the following properties:*

- a. Then the degree is monotone; for all $a \in \mathcal{T}$, if a^C is a child of a , then $\deg a^C \leq \deg a$.
- b. We have $\deg a_0 = \deg a_{-l}$, for all $l \geq 1$.
- c. We have $\deg a_0 > \deg a$, for all $a \in \mathcal{T}_l$ with $l \geq 1$.

Throughout this paper, let $\deg a_0 = d$. We say that \mathcal{T} is a tree with dynamics of degree d .

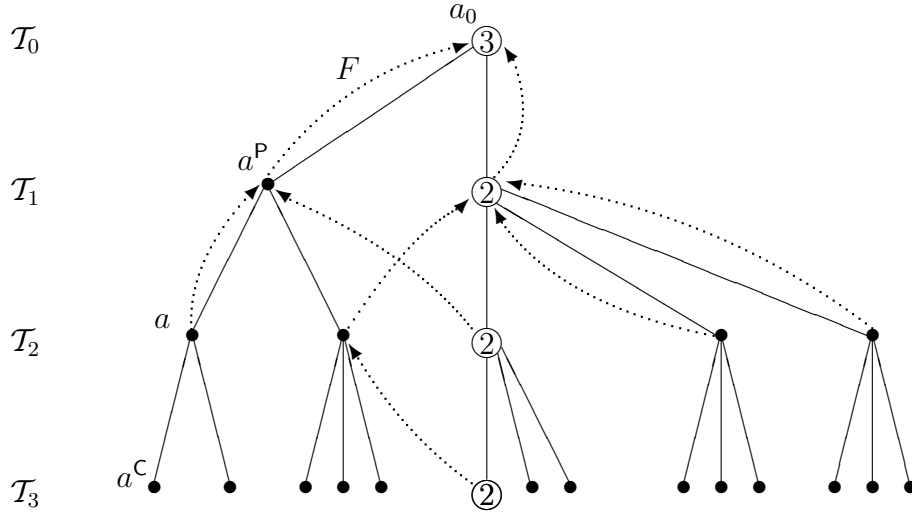


Figure 4: A tree \mathcal{T} with dynamics F of degree 3. For clarity, most of the dynamics from level 3 and the extended root are not shown.

The dynamics are a d -fold branched cover of \mathcal{T} by itself.

Lemma 4.5. [E, Lem. 4.11] *Any vertex of \mathcal{T} has exactly d pre-images under F , counted by degree. That is, for any $a \in \mathcal{T}$,*

$$\sum_{\{b \in F^{-1}(a)\}} \deg b = d.$$

We consider all infinite geodesics from the root that move down the tree.

Definitions 4.6. *An end of \mathcal{T} is a sequence $\vec{x} = (x_l)_{l=0}^\infty$, where $x_l \in \mathcal{T}_l$ and x_{l+1} is a child of x_l for all l . Note that a children preserving map takes ends to ends, so $F(\vec{x})$ is well defined. We define the degree of an end $\vec{x} = (x_l)_{l=0}^\infty$, by $\deg \vec{x} = \lim_{l \rightarrow \infty} \deg x_l$. If $\deg \vec{x} > 1$, then \vec{x} is called a critical end. Let \mathcal{B} denote the set of all ends of \mathcal{T} . We call \mathcal{B} the end space of \mathcal{T} .*

We can define an ultra-metric on \mathcal{B} by

$$\text{dist}(\vec{x}, \vec{y}) = e^{-L}, \quad \text{where } L = \sup \{l : x_l = y_l\},$$

for $\vec{x} \neq \vec{y}$, and $\text{dist}(\vec{x}, \vec{x}) = 0$. With this metric, \mathcal{B} is a Cantor set. Since F is children-preserving, it extends to a continuous map on \mathcal{B} . This metric restricts to a metric on \mathcal{T} , and the end space can naturally be regarded as the boundary of \mathcal{T} in this topology.

We can extend τ in a natural way to a map from \mathcal{B} to \mathcal{K}_f / \sim , where $z_1 \sim z_2$ if z_1 and z_2 are in the same connected component of \mathcal{K}_f . Recall, that if A is an annulus, then $P(A)$ is the filled-in annulus: $A \cup \{\text{bounded components of } \mathbb{C} \setminus A\}$.

Definition 4.7. Let $\vec{x} = (x_l)_{l=0}^\infty \in \mathcal{B}$. For each l , $\tau(x_l)$ is an annulus of f . Define $\tau : \mathcal{B} \rightarrow \mathcal{K}_f / \sim$ by

$$\tau(\vec{x}) = \bigcap_{l=0}^{\infty} P(\tau(x_l)).$$

Proposition 4.8. The map $\tau : \mathcal{B} \rightarrow \mathcal{K}_f / \sim$ is a homeomorphism.

Proof. A Cantor set is homeomorphic to an inverse limit system given by a sequence of non-trivial open/closed partitions of itself, where the partition at stage $l+1$ is a refinement of the partition at stage l [HY, Thm. 2–96]. For \mathcal{B} one such inverse limit system is given by

$$\begin{aligned} \mathcal{T}_l &\leftarrow \mathcal{T}_{l+1} \\ a^P &\leftarrow a. \end{aligned}$$

For \mathcal{K} / \sim the equivalent inverse limit system is given by

$$\begin{aligned} U_l &\leftarrow U_{l+1} \\ A^P &\leftarrow A, \end{aligned}$$

where A^P is the unique annulus of f at level l that A is nested inside.

By definition, τ induces an isomorphism of these inverse limit systems. Therefore it is a homeomorphism. □

4.2 Combinatorial Harmonic Measure

Definition 4.9. Let \mathcal{T} be a tree with a distinguished root. A flow on \mathcal{T} is a function $\Omega : \mathcal{T} \rightarrow [0, \infty)$ such that for all $a \in \mathcal{T}$ we have

$$\Omega(a) = \sum_{\{a^c\}} \Omega(a^c).$$

It is well known that flows on \mathcal{T} are in one-to-one correspondence with finite measures on \mathcal{B} . In Theorem 4.18 we outline the proof of this fact.

A useful way to think of a flow is in terms of electrical networks [DS]. Imagine the tree is an electrical network, grounded at its ends, and a charge is introduced at the root. The electricity will flow from the root to the ends. For each $a \in \mathcal{T}$, $\Omega(a)$ is current that flows through a . Equivalently, one could imagine that the edge from a^P to a is a wire, and $\Omega(a)$ is its conductance. The total charge on a set of ends is the measure of the set.

We define Ω , a combinatorial version of harmonic measure. Intuitively, the measure of a is distributed to the d pre-images of a . Each pre-image receives an amount of measure proportional to its degree. Compare to the Brolin Measure [Bro, §16].

Definition 4.10. For $a \in \mathcal{T}$, define $\Omega(a)$ by $\Omega(a) = 1$ if a is in the extended root of \mathcal{T} , and

$$\Omega(a) = \frac{\deg a}{d} \Omega(F(a))$$

otherwise. We call Ω the combinatorial harmonic measure of \mathcal{T} .

Although we refer to Ω as a “measure,” at this moment $\Omega(a)$ is just a *weight*—a number associated to each $a \in \mathcal{T}$. It is not clear that it is a flow, and since the proof is rather technical we defer it. Nonetheless, we can use Ω to estimate $\omega_f(A)$. We prove Theorem 4.12, which implies Theorem 3.3 and thus most of the results in Section 3. We then show that Ω is a flow. We outline the extension of Ω to a measure on \mathcal{B} . Finally, we show that τ is a measure isomorphism between (\mathcal{B}, Ω) and (\mathcal{K}, ω_f) .

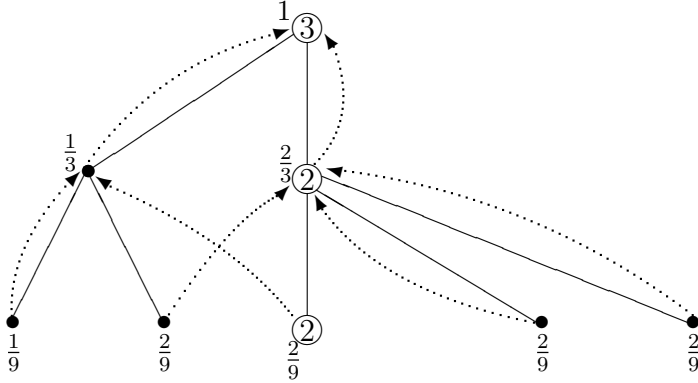


Figure 5: A tree with dynamics with the measure $\Omega(a)$ shown.

It is worth noting that by Lemma 4.5, Ω is F -invariant. In the sense that for any $a \in \mathcal{T}$,

$$\Omega(F^{-1}(a)) = \Omega(a).$$

Although, we will not use this fact in this paper.

Lemma 4.11. If $a \in \mathcal{T}_l$ for some $l \geq 0$, then

$$\Omega(a) = d^{-[l/H]} \prod_{n=0}^{[l/H]} \deg F^n(a).$$

Proof. By definition of Ω , we have

$$\Omega(a) = \frac{\deg a}{d} \Omega(F(a)) \tag{1}$$

$$= \frac{\deg a}{d} \frac{\deg F(a)}{d} \Omega(F^2(a)) \tag{2}$$

$$= d^{-2} \deg a \deg F(a) \Omega(F^2(a)). \tag{3}$$

Say that $l = kH + h$, for $0 \leq k$ and $0 < h \leq H$. Then, $\lceil l/H \rceil = k + 1$, so $F^{k+1}(a) = a_{h-H}$. Repeat the above argument k times. \square

The measure decreases exponentially with the level of the tree. Note that this is a combinatorial version of Theorem 3.3.

Theorem 4.12. *Let $D = \max \deg \vec{x}$, for ends $\vec{x} \in \mathcal{B}$. There exists a constant c_0 such that for all $l \geq 0$, if $a \in \mathcal{T}_l$, then*

$$\Omega(a) \leq c_0 \left(\frac{D}{d} \right)^{\lceil l/H \rceil}.$$

Proof. There are a finite number of levels of the tree that contain a vertex b with $\deg b > D$, say Q of them, and let $q = \lceil Q/H \rceil$. Define $c_0 = (d/D)^q$. Because the dynamics go up H levels, the iterates of a point can hit at most n of the levels with a vertex of high degree. Hence with at most q exceptions, we have $\deg F^n(a) \leq D$, so we can replace those terms in Lemma 4.11 with D . For the exceptional iterates $\deg F^n(a) \leq d$, and c_0 was defined in such a way to reflect this. \square

The above estimate is sharp when there is an end \vec{y} with $\deg \vec{y} = D$ and $F(\vec{y}) = \vec{y}$. In general, we can get a little better estimate for a particular end.

Corollary 4.13. *Let $\vec{x} \in \mathcal{B}$, then there exists a constant $c = c(\vec{x})$, such that*

$$\Omega(x_l) \leq c \left(\frac{\deg \vec{x}}{d} \right)^{\lceil l/H \rceil}.$$

Proof. Similar to the above lemma. The key difference is that for l sufficiently large, we have $\deg x_l = \deg \vec{x}$. \square

Corollary 4.14. *Let $\vec{x} \in \mathcal{B}$, then*

$$\lim_{l \rightarrow \infty} \Omega(x_l) = 0.$$

Proof. Note that $\deg \vec{x} < d$ by Lemma 4.4.3 and apply the above corollary. \square

We now establish the first part of the correspondence between the measures on the tree and in the plane.

Lemma 4.15. *For all $a \in \mathcal{T}$,*

$$\Omega(a) = \omega_f(\tau(a)).$$

Proof. Suppose that $a \in \mathcal{T}_l$ and use induction on l . If $l \leq 0$, there is only one annulus at level l , so $P(\tau(a)) \cap \mathcal{K}_f = \mathcal{K}_f$ and $\Omega(a) = 1 = \omega_f(\tau(a))$. For $l > 0$, let $\tau(a) = A$. We compute $\omega_f(A)$ in terms of $\omega_f(f(A))$. Now f expands arcs of the circle by a factor of d , so we need to multiply $\omega_f(A)$ by d . However, $f|A$ is a $(\deg f|A)$ -to-one map, so we must

divide by $\deg f|A$. Therefore, $\omega_f(f(A)) = (d/\deg f|A)\omega_f(a)$. Note that $F(a) \in \mathcal{T}_{l-H}$, so the inductive hypothesis applies to $F(a)$. It follows that

$$\begin{aligned}
 \Omega(a) &= \frac{\deg a}{d} \Omega(F(a)) && \text{by Definition 4.10,} \\
 &= \frac{\deg a}{d} \omega_f(\tau(F(a))) && \text{by induction,} \\
 &= \frac{\deg f|A}{d} \omega_f(f(\tau(a))) && \text{by definitions of } \deg a \text{ and } F, \\
 &= \omega_f(\tau(a)).
 \end{aligned}$$

□

Therefore, we can transfer the estimate from Theorem 4.12 to a nested sequence of annuli of f . Which is exactly the content of Theorem 3.3. Therefore, all results in Section 3 have now been proven.

4.3 Extending the Measure to the End Space

We show that Ω is a flow. That is, measure is inherited by children, as well as pre-images.

Lemma 4.16. *For all $a \in \mathcal{T}$,*

$$\Omega(a) = \sum_{\{a^c\}} \Omega(a^c).$$

Proof. Let $a \in \mathcal{T}_l$. If $l < 0$, it is clear. For $l = 0$, that is $a = a_0$, we have $\{a_0^c\} = F^{-1}(a_{-H+1})$, since $\mathcal{T}_1 = \{a_0^c\}$. By Lemma 4.5,

$$\sum_{\{a \in F^{-1}(a_{-H+1})\}} \deg a = d.$$

Recall that $\Omega(a_0) = \Omega(a_{-H+1}) = 1$. We have

$$\begin{aligned}
 \Omega(a_0) &= 1 \\
 &= \frac{1}{d} \sum_{\{a \in F^{-1}(a_{-H+1})\}} \deg a \\
 &= \sum_{\{a_0^c\}} \frac{\deg a_0^c}{d} \Omega(a_{-H+1}) \\
 &= \sum_{\{a_0^c\}} \frac{\deg a_0^c}{d} \Omega(F(a_0^c)) \\
 &= \sum_{\{a_0^c\}} \Omega(a_0^c).
 \end{aligned}$$

We use induction on $l > 0$. Let $a \in \mathcal{T}_l$ and $F(a) = b$. Note that $b \in \mathcal{T}_{l-H}$, so $\Omega(b) = \sum \Omega(b^{c_j})$, by the inductive hypothesis.

$$\begin{aligned} \Omega(a) &= \Omega(b) \frac{\deg a}{d} \\ &= \frac{1}{d} \sum_{\{b^{c_j}\}} \Omega(b^{c_j}) \deg a, \quad \text{by induction} \\ &= \frac{1}{d} \sum_{\{b^{c_j}\}} \Omega(b^{c_j}) \sum_{\{a^{c_i} \in F^{-1}(b^{c_j})\}} \deg a^{c_i}, \end{aligned}$$

by 4.4.c applied to each child b^{c_j} of b ,

$$\begin{aligned} &= \sum_{\{a^{c_i}\}} \frac{\Omega(F(a^{c_i}))}{d} \deg a^{c_i}, \quad \text{since } F(\{a^{c_i}\}) = \{b^{c_j}\} \\ &= \sum_{\{a^{c_i}\}} \Omega(a^{c_i}). \end{aligned}$$

□

Definitions 4.17. For $a \in \mathcal{T}$, we define \mathcal{U}_a , the cone of a , as the set of all ends that pass through a . That is,

$$\mathcal{U}_a = \{\vec{x} \in \mathcal{B} : a \in \vec{x}\}.$$

Define the measure of the cone of a by

$$\Omega(\mathcal{U}_a) = \Omega(a).$$

For any $a \in \mathcal{T}$, \mathcal{U}_a is an open ball in \mathcal{B} ; it is also compact. The set of all cones is a sub-basis for the topology of \mathcal{B} . Moreover, it is an algebra.

Following Cartier, we outline the proof that Ω is to a measure on \mathcal{B} .

Theorem 4.18. [C, Thm. 2.1] We can extend Ω to a complete Borel measure on \mathcal{B} . We call Ω the combinatorial harmonic measure on \mathcal{B} .

Proof. It follows from Lemma 4.16 that Ω is finitely additive on cones. Thus, Ω is a pre-measure. By standard techniques (Carathéodory's Theorem), we can extend Ω to an outer measure and then a measure on \mathcal{B} . □

In general, a measure induced by a flow is called a harmonic measure. A tree with dynamics has a preferred harmonic measure.

We now prove that the harmonic measure on the tree is isomorphic to the harmonic measure in the plane.

We restate Theorem B in more detail.

Theorem B. *Let f be a polynomial with disconnected Julia set. The harmonic measure ω_f and the combinatorial harmonic measure Ω are isomorphic. Moreover, τ induces a measure isomorphism of (\mathcal{B}, Ω) and $(\mathcal{K}_f, \omega_f)$.*

Proof. Let $z_1 \sim z_2$ if they are in the same component of \mathcal{K}_f . Let $\pi : \mathcal{K}_f \rightarrow \mathcal{K}_f / \sim$ be the projection map. We can consider $\pi^* \omega_f$, the push-forward by the projection map of ω_f . By Theorem A, π is a bijection, except on a set of measure zero. Thus, for all $X \subset \mathcal{K}_f$ measurable, we have

$$\omega_f(X) = \omega_f(\pi^{-1}\pi(X)).$$

That is, π is a measure isomorphism between (\mathcal{K}, ω_f) and $(\mathcal{K} / \sim, \pi^* \omega_f)$.

We can also consider $\tau^* \Omega$ the push-forward of Ω defined on \mathcal{K} / \sim . We have two measures defined on \mathcal{K}_f / \sim , we show that they are equal. By Proposition 4.8, τ is a homeomorphism. By Lemma 4.15, $\pi^* \omega_f$ and $\tau^* \Omega$ agree on a sub-basis for the topology of \mathcal{K}_f . Therefore, they are equal. That is, if $\mathcal{X} \subset \mathcal{B}$ is measurable, then

$$\Omega(\mathcal{X}) = \omega_f(\pi^{-1}\tau(\mathcal{X})).$$

□

This isomorphism gives a new method to compute the harmonic measure of subsets of \mathcal{K}_f using annuli of f .

Theorem 4.19. *Let \mathcal{K}_f be a disconnected polynomial Julia set. Let X be a measurable subset of \mathcal{K}_f . For $l \geq 0$, let $\{A_{l,1}, \dots, A_{l,I(l)}\}$ be the annuli of f at level l such that $X \cap P(A_{l,i}) \neq \emptyset$ for $i = 1, \dots, I(l)$. Then*

$$\omega_f(X) = \lim_{l \rightarrow \infty} \sum_{i=1}^{I(l)} \omega_f(A_{l,i}).$$

Proof. The inequality

$$\omega_f(X) \leq \lim_{l \rightarrow \infty} \sum_{i=1}^{I(l)} \omega_f(A_{l,i}),$$

follows easily from Lemma 3.2. The opposite inequality is not clear. We can consider, $\mathcal{X} = \tau^{-1}(X)$ and show the analogous inequality for Ω . Filled-in annuli are analogous to cones. So what we want to show is

$$\Omega(\mathcal{X}) = \lim_{l \rightarrow \infty} \sum_{i=1}^{I(l)} \Omega(a_{l,i}),$$

where $\mathcal{X} \cap \mathcal{U}_{a_{l,i}} \neq \emptyset$.

We may assume that \mathcal{X} is compact, since Ω is Borel. Fix $\varepsilon > 0$. We can find $\mathcal{V} \subset \mathcal{B}$ open, such that $\mathcal{X} \subset \mathcal{V}$ and $\Omega(\mathcal{V}) \leq \Omega(\mathcal{X}) + \varepsilon$. Cones are open balls in \mathcal{B} and \mathcal{X} is compact, so we can find finitely many cones, $\mathcal{U}_{b_1}, \dots, \mathcal{U}_{b_J}$, such that

$$\mathcal{X} \subset \bigcup \mathcal{U}_{b_j} \subset \mathcal{V}.$$

By replacing b_j by $\{b_j^C\}$, several times if necessary, we may assume that there is an L , such that $b_j \in \mathcal{T}_L$ for all j . Hence, $\{a_{L,i}\} \subset \{b_j\}$. Therefore,

$$\sum_{i=1}^{I(L)} \Omega(a_{L,i}) \leq \sum_{j=1}^J \Omega(b_j) \leq \Omega(\mathcal{V}) \leq \Omega(\mathcal{X}) + \varepsilon.$$

□

Just as the harmonic measure in the plane can be defined in terms of Brownian particles, the combinatorial harmonic measure can be defined in terms of a random walk. A *random walk* on a tree is a discrete time Markov chain on the tree. We imagine a particle moving around the tree, with the Markov chain describing its position at each time. A random walk is defined its transition function $\text{tran}(x, y)$, which gives the probability that a particle in x will move to y . A *nearest neighbor* random walk is a random walk where $\text{tran}(x, y) = 0$, unless x is adjacent to y . A flow induces a nearest neighbor random walk. The nearest neighbor random walk induced by Ω is the following.

Definition 4.20. Define a (nearest neighbor) random walk on \mathcal{T} by

$$\text{tran}(a, a^P) = \frac{1}{2}, \quad \text{tran}(a, a^C) = \frac{1}{2} \frac{\Omega(a^C)}{\Omega(a)},$$

and define all other transition probabilities to be zero.

It is straightforward to show that the above random walk almost surely hits the end space. That is, it is *transitive*, it almost surely visits a given vertex a finite number of times. A *loop-erased* random walk is a random walk without repeated vertices. We can transform the above random walk into a loop-erased random walk. We define

$$\text{tran}(a, a^C) = \frac{\Omega(a^C)}{\Omega(a)},$$

and all other transition probabilities are zero. Effectively, this gives a random end $\vec{x} \in \mathcal{B}$. The above random walks correspond to the combinatorial harmonic measure. Given $\mathcal{X} \subset \mathcal{B}$ (measurable), $\Omega(\mathcal{X})$ is the probability that a (loop-erased) random walk hits \mathcal{X} .

This can be regarded as a combinatorial version of Brownian motion in the plane. Suppose a Brownian particle starts in U_0 and moves randomly in the plane. It is known that the hitting probability of this Brownian path is still the harmonic measure [La, Prop. 9]. Thus, we almost surely obtain a sequence $(A_i)_{i=0}^\infty$ of annuli of f , where $A_0 = U_0$ and A_{i+1} is the first annulus of f visited by the Brownian path after A_i . We call this sequence an *itinerary* of the Brownian path. We can also obtain a *loop-erased* itinerary by deleting repetition. That is, an itinerary $(A_l)_{l=1}^\infty$, where A_l is an annulus at level l . The probability that a given set of itineraries occurring is equal to the measure of the analogous set of ends.

References

- [A] Maynard Arsove, *Green's measure and harmonic measure*, Michigan Math. J. **19** (1972), 275–276. [MR 0304682](#)
- [B] Bodil Branner, *Puzzles and para-puzzles of quadratic and cubic polynomials*, Complex Dynamical Systems (Robert L. Devaney, ed.), Proc. Sympos. Appl. Math., vol. 49, AMS, 1994, pp. 31–67. [MR 1315533](#)
- [BH] Bodil Branner and John Hubbard, *Iteration of cubic polynomials, part II: Patterns and parapatterns*, Acta Math. **169** (1992), 229–325. [MR 1194004](#)
- [BreC] M. Brelot and G. Choquet, *Espaces et lignes de Green*, Ann. Inst. Fourier Grenoble **3** (1951), 199–263. [MR 0062883](#)
- [Bro] Hans Brolin, *Invariant sets under iteration of rational functions*, Ark. Mat. **6** (1965), 103–144 (1965). [MR 0194595](#)
- [C] P. Cartier, *Fonctions harmoniques sur un arbre*, Symposia Math. **9** (1972), 203–270. [MR 0353467](#)
- [DeMc] Laura DeMarco and Curtis T. McMullen, *Trees and the dynamics of polynomials*, Pre-print, 8/2006. Available at <http://www.arxiv.org/abs/math.DS/0608759>.
- [Do] J. L. Doob, *Classical Potential Theory and Its Probabilistic Counterpart*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 262, Springer-Verlag, New York, 1984. [MR 731258](#)
- [DS] P. G. Doyle and J. L. Snell, *Random Walks and Electrical Networks*, MAA, Washington, 1984. [MR 920811](#)
- [E] Nathaniel D. Emerson, *Dynamics of polynomials with disconnected Julia sets*, Discrete Contin. Dyn. Syst. **9** (2003), no. 4, 801–834. [MR 1975358](#)
- [HY] John G. Hocking and Gail S. Young, *Topology*, Addison-Wesley, 1961. [MR 1016814](#)
- [JW] Peter W. Jones and Thomas H. Wolff, *Hausdorff dimension of harmonic measures in the plane*, Acta. Math. **161** (1988), 131–144. [MR 962097](#)
- [La] Steven P. Lalley, *Brownian motion and the equilibrium measure on the Julia set of a rational mapping*, Ann. Probab. **20** (1992), no. 4, 1932–1967. [MR 1188049](#)
- [Lo] Artur O. Lopes, *An analogy of the charge distribution on Julia sets with the Brownian motion*, J. Math. Phys. **30** (1989), no. 9, 2120–2124. [MR 1009928](#)
- [LP] G. Levin and F. Przytycki, *External rays to periodic points*, Israel J. Math. **94** (1996), 29–57. [MR 1394566](#)

- [Ø1] Bernt Øksendal, *Brownian motion and sets of harmonic measure zero*, Pacific J. Math. **95** (1981), no. 1, 179–192. [MR 631668](#)
- [QY] Weiyuan Qiu and Yongcheng Yin, *A proof of the Branner-Hubabrd conjecture on Cantor Julia sets*, Pre-print, 2006. Available at <http://www.arxiv.org/abs/math.DS/0608045>.
- [R] Thomas Ransford, *Potential Theory in the Complex Plane*, Cambridge UP, 1995. [MR 1334766](#)
- [S] Norbert Steinmetz, *Rational Iteration*, De Gruyter, 1993. [MR 1224235](#)

University of Southern California
Los Angeles, CA 90089

E-mail: nemerson@usc.edu